MA3110S

Basics

Norm: nonnegative $\|\vec{v}\| \ge 0$

scalable $\|\alpha \vec{v}\| = \alpha \|\vec{v}\|$

Δ ineq: $\|\vec{v}\| + \|\vec{w}\| \ge \|\vec{v} + \vec{w}\|$

Open: $\forall a \in S, \exists r > 0$ s.t. $B(a,r) \subseteq S$ ← open ball

Closed: $\mathbb{R}^n \setminus S$ is open

Compact: Every open cover has finite subcover.

Seq. compact: Every sequence has a subsequence converging to some $x \in S$

Bounded: $\exists M \in \mathbb{R}$ s.t. $\forall x \in S$, $\|x\| \le M$

Bolzano-Weierstrass: Every bounded sequence in $\mathbb{R}^n$ has a convergent subseq.

Cauchy equiv: In $\mathbb{R}^n$, convergence ⟺ Cauchy

CBC: In $\mathbb{R}^n$, cpt ⟺ seq. cpt ⟺ closed+bdd.

Lebesgue covering: In $\mathbb{R}^n$, if S is closed+bdd and S is covered by open sets, then $\exists \delta > 0$ s.t. $\forall x \in S$, $B(x,r)$ is in some set.

Continuous: $\forall a \in S$, $\lim_{\substack{x \to a \\ x \in S}} f(x) = f(a)$

Equiv cts: ∀ open sets $V \in \mathbb{R}^m$, $f^{-1}(V)$ is open rel to S

Cts transitivity: f, g cts $\Rightarrow g \circ f$ cts

Cts cpt: f cts $\Rightarrow (S$ cpt $\Rightarrow f(S)$ cpt$)$

Disconnected: ∃ disjoint open sets U, V s.t. $S \subset U \cup V$, $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$

Path-connected: $\forall x, y \in S$, ∃ cts path $\gamma \in S$ s.t. $\gamma(0) = x$, $\gamma(1) = y$.

PCC: Path-connected ⇒ Connected

Cts connected: f cts $\Rightarrow (S$ connected $\Rightarrow f(S)$ connected$)$

Cts path-connected: f cts $\Rightarrow (S$ path-connected $\Rightarrow f(S)$ path-connected$)$

Differentiation

Differentiability: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = 0$

Diff. cts.: Differentiable ⇒ Cts

Mean val. thm.: If $a, b \in \mathbb{R}$, $a < b$, f cts on $[a,b]$, f diffable on (a,b) then $\exists c \in (a,b)$ s.t. $f(b) - f(a) = f'(c)(b-a)$

Cauchy mean val. thm.: If $a, b \in \mathbb{R}$, $a < b$, f, g cts on $[a,b]$, f, g diffable on (a,b) then $\exists c \in (a,b)$ st. $g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$

L'Hopital (1): Suppose $f, g : [a,b] \to \mathbb{R}$, f, g cts on $[a,b]$, f, g diffable on (a,b), $f(a) = 0 = g(a)$, $g(x), g'(x) \neq 0$ $\forall x \in (a,b]$. Then, if $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists then $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ exists and takes the same value.

L'Hopital (2): Suppose $f, g : [a,b] \to \mathbb{R}$, f, g diffable on (a,b), $\lim_{x \to a^+} f(x) = \infty = \lim_{x \to a^+} g(x)$. Then, if $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists then $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ exists and takes the same value.

L'Hopital (Remark): L'Hopital works for both left & right limits.

Intermediate val. thm.: If $a, b \in \mathbb{R}$, $a < b$, f cts on $[a,b]$, then: $\forall u \in (\min\{f(a), f(b)\}, \max\{f(a), f(b)\})$, $\exists c \in (a,b)$ s.t. $f(c) = u$.

Intermediate val. thm. of derivative: If $a, b \in \mathbb{R}$, $a < b$, f cts on $[a,b]$, f diffable on (a,b), then: f' satisfies IVT, i.e. $\forall \lambda : f'(t_1) < \lambda < f'(t_2)$, $\exists c \in (t_1, t_2)$ s.t. $f'(c) = \lambda$.

Taylor's thm: If $x, b \in \mathbb{R}$, $x < b$, U is open, $f : U \to \mathbb{R}$, $f^{(n)}$ cts on $[x,b]$, $f^{(n+1)}$ exists on (x,b), then $\exists c \in (x,b)$ s.t.

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-x)^{n+1}$$

(the summation is the n-th degree Taylor polynomial of f at x)

Local minimum: Point $a \in S$ where $\exists r > 0$ s.t. $B := B(a,r) \subseteq S$ (i.e. r is an interior pt) and $\forall x \in B$, $f(x) \ge f(a)$.

Local min/max from derivative: Suppose that U is an open subset of $\mathbb{R}$ and $f : U \to \mathbb{R}$ and $f^{(n+1)}$ cts on U, and given $a \in U$, $f'(a) = \dots = f^{(n)}(a) = 0$, $f^{(n+1)}(a) \neq 0$. Then:

① If n even, then a is neither local min nor local max.

② If n odd and $f^{(n+1)}(a) > 0$, then a is local min.

③ If n odd and $f^{(n+1)}(a) < 0$, then a is local max.

Integration

Partition: finite subset of $[a,b]$ containing the endpoints: $a = x_0 < x_1 < x_2 < \dots < x_p = b$

Mesh: size of largest interval in partition: $|P| := \max_{1 \le k \le p} (x_k - x_{k-1})$

Riemann sum: $R(f,P) := \sum_{k=1}^{p} f(t_k)(x_k - x_{k-1})$ where t_k is any value in $[x_{k-1}, x_k]$

Riemann integrability: $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall P : |P| \le \delta$, $\forall t_k \in [x_{k-1}, x_k]$, $|R(f,P) - L| \le \varepsilon$. Then $\int_a^b f\,dx = L$, and $f \in \mathcal{R}[a,b]$.

Equiv. Cauchy for integrability: $f \in \mathcal{R}[a,b] \iff \forall \varepsilon > 0, \exists \delta > 0$, $\forall P, Q : |P|, |Q| \le \varepsilon$, $\forall t_{k,P} \in [x_{k-1,P}, x_{k,P}]$, $\forall t_{k,Q} \in [x_{k-1,Q}, x_{k,Q}]$, $|R(f,P) - R(f,Q)| \le \varepsilon$

Integrable ⇒ Bounded: All functions in $\mathcal{R}[a,b]$ are bounded.

Upper sum: $U(f,P) := \sum_{k=1}^{p} M_k (x_k - x_{k-1})$ where $M_k := \sup_{t \in [x_{k-1}, x_k]} f(t)$

Lower sum: $L(f,P) := \sum_{k=1}^{p} m_k (x_k - x_{k-1})$ where $m_k := \inf_{t \in [x_{k-1}, x_k]} f(t)$

Upper & lower squeeze: $f \in \mathcal{R}[a,b] \iff \forall \varepsilon > 0, \exists \delta > 0, \forall P : |P| \le \delta$, $U(f,P) - L(f,P) \le \varepsilon$

Integrability by existence of partition: $f \in \mathcal{R}[a,b] \iff \forall \varepsilon > 0$, $\exists P$ st. $U(f,P) - L(f,P) \le \varepsilon$

g-null/measure zero: $S \subseteq [a,b]$ where $\forall \varepsilon > 0$, $\exists ([c_k, d_k])_{k=1}^{\infty} \subseteq [a,b]$ s.t. $S \subseteq \bigcup_{k=1}^{\infty} [c_k, d_k]$ and $\sum_{k=1}^{\infty} (d_k - c_k) < \varepsilon$

Equiv. integrability for discontinuous f: Suppose f bounded. Then $f \in \mathcal{R}[a,b] \iff$ set of discontinuities of f has measure zero.

Continuous ⇒ Integrable: f cts on $[a,b] \Rightarrow f \in \mathcal{R}[a,b]$

Discontinuities of monotonic f at most countable.

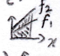
Monotonic ⇒ Integrable: Follows from above.

Properties of integral:

① $\int_a^b (\alpha f_1 + f_2)\,dx = \alpha \int_a^b f_1\,dx + \int_a^b f_2\,dx$ (i.e. $\mathcal{R}[a,b]$ is a vector space)

② If $a < c < b$ then $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$

③ If $f_1 \le f_2$ then $\int_a^b f_1\,dx \le \int_a^b f_2\,dx$



④ If $|f| \le M$ then $\left|\int_a^b f\,dx\right| \le M(b-a)$

Integrability-continuity transitivity: If $f \in \mathcal{R}[a,b]$, $m \le f \le M$, ϕ cts on $[m,M]$, $h(x) := \phi(f(x))$, then $h \in \mathcal{R}[a,b]$.

Properties due to transitivity:

① $f, g \in \mathcal{R}[a,b] \Rightarrow fg \in \mathcal{R}[a,b]$

② $f \in \mathcal{R}[a,b] \Rightarrow |f| \in \mathcal{R}[a,b]$ and $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$

Integration by parts: $\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df$.

Change of variables: Suppose φ strictly increasing, $\varphi(A) = a$, $\varphi(B) = b$, $f \in \mathcal{R}[a,b]$, then: $f \circ \varphi \in \mathcal{R}(\varphi, [a,b])$ and $\int_a^b f\,dx = \int_A^B f(\varphi(y))\,d\varphi = \int_A^B f(\varphi(y))\varphi'(y)\,dy$

Fundamental theorem of calculus I: If $f \in \mathcal{R}[a,b]$ and $F(x) := \int_a^x f\,dx$ then: F cts on $[a,b]$; if f cts at $c \in [a,b]$ then F diffable at c and $F'(c) = f(c)$.

Fundamental theorem of calculus II: If $f \in \mathcal{R}[a,b]$, F diffable on $[a,b]$, $F'(x) = f(x)$ $\forall x \in [a,b]$, then: $\int_a^b f\,dx = F(b) - F(a)$.

Integration by parts: Suppose F, G diffable on $[a,b]$, $F' = f \in \mathcal{R}[a,b]$, $G' = g \in \mathcal{R}[a,b]$, then: $\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$.

Sequences & Series

Pointwise convergence: $\forall x \in E$, $\lim_{n \to \infty} f_n(x) = f(x)$.

Uniform convergence: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \ge N$, $\forall x \in E$, $|f_n(x) - f(x)| < \varepsilon$

Supremum gap: If $f_n \xrightarrow{\text{ptwise}} f$ and $M_n := \sup_{x \in E} |f_n(x) - f(x)|$ then: $f_n \xrightarrow{\text{unif}} f \iff \lim_{n \to \infty} M_n = 0$

Equiv. Cauchy for unif. conv.: $f_n \xrightarrow{\text{unif.}} f \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \ge N$, $\forall x \in E$, $|f_n(x) - f_m(x)| < \varepsilon$

Weierstrass test for unif. conv. of series: Suppose $|f_n| \le M_n$, then: $\sum_{n=1}^{\infty} M_n$ converges $\Rightarrow \sum_{n=1}^{\infty} f_n$ unif. converges

Swapping limits: If $f_n \xrightarrow{\text{unif}} f$, a is limit pt of E, then $\lim_{x \to a} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to a} f_n(x)$ (in particular, this means that $\{\lim_{x \to a} f_n(x)\}$ conv.)

Continuous limit to continuous: If $\{f_n\}$ cts, $f_n \xrightarrow{\text{unif}} f$, then f cts.

Extending pointwise to uniform convergence: Suppose K is compact. If $\{f_n\}$ cts on K, $f_n \xrightarrow{\text{ptwise}} f$, $f_n \ge f_{n+1}$ on K, then $f_n \xrightarrow{\text{unif}} f$.

Complete metric space: Every Cauchy seq. has a limit pt.

Space of cts functions is complete: The space of cts, bounded functions on $\mathbb{R}$, where $d(f,g) := \sup_{x \in \mathbb{R}} |f(x) - g(x)|$, is complete.

Uniform convergence & differentiation: If $\{f_n\}$ are diffable, f_n' unif. conv. on $[a,b]$, $\exists x_0 \in [a,b]$ s.t. $f_n(x_0)$ converges, then: $\exists f : f_n \xrightarrow{\text{unif}} f$, $\lim_{n \to \infty} f_n'(x) = f'(x)$.

Uniform convergence & integration: If $f_n \in \mathcal{R}[a,b]$ and $f_n \xrightarrow{\text{unif}} f$, then $f \in \mathcal{R}[a,b]$ and $\lim_{n \to \infty} \int_a^b f_n\,dx = \int_a^b f\,dx$

Swapping integral with series of unif. conv. functions: If $f_n \in \mathcal{R}[a,b]$ and $\sum_{n=1}^{\infty} f_n \xrightarrow{\text{unif.}} f$, then $\int_a^b f\,dx = \sum_{n=1}^{\infty} \int_a^b f_n\,dx$

Pointwise bounded: $\exists g : E \to \mathbb{R}$ s.t. $\forall x \in E, \forall n \in \mathbb{N}$, $f_n(x) \le g(x)$

Uniformly bounded: $\exists M \in \mathbb{R}$ s.t. $\forall x \in E, \forall n \in \mathbb{N}$, $f_n(x) \le M$.

Equicontinuous: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall n \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$ where $d(x,y) < \delta$, $|f_n(x) - f_n(y)| < \varepsilon$ (can also be extended to uncountable families of fns.)

Continuous, Compact, Unif. conv, equicts: If K is compact, $\{f_n\}$ are cts, $f_n \xrightarrow{\text{unif}} f$, then $\{f_n\}$ is equicontinuous.

Containing unif. conv. subsequence: If K is compact, $\{f_n\}$ are cts, $\{f_n\}$ ptwise bounded, $\{f_n\}$ equicontinuous, then $\{f_n\}$ uniformly bounded and contains unif. conv. subseq.

$C(X)$: space of all complex-valued, cts, bdd functions on metric space X.

Normed vector space: vector space with norm.

Normed vector space ⇒ metric space with $d(x,y) := \|x - y\|$

Banach space: A normed space whose induced metric space is complete.

Algebra: A vector space equipped add^n with vec. product: $f(g+h) = fg + fh$ $\quad (f+g)h = fh + gh$ $\quad c(fg) = f(cg)$

$C(X)$ is an algebra with the function product.

Banach algebra: A Banach space that is an algebra satisfying $\|fg\| \le \|f\|\|g\|$.

$C(X)$ is a Banach algebra.

Weierstrass: For any cts function f on $[a,b]$, ∃ polynomials P_n s.t. $P_n \xrightarrow{\text{unif}} f$ on $[a,b]$. (e.g. using the Bernstein polynomials)

Separates points: Given a family of functions $A \subseteq C(X)$. A separates points iff $\forall x_1 \neq x_2 \in X$, $\exists f \in A$ s.t. $f(x_1) \neq f(x_2)$

Vanishes at no point: $\forall x \in X$, $\exists f \in A$ s.t. $f(x) \neq 0$.

Stone-Weierstrass: If X is compact and $A \subseteq C(X)$, A separates points, A vanishes at no point, then A is dense in $C(X)$, i.e. $\bar{A} = C(X)$ unif.

Power series: $\sum_{n=0}^{\infty} c_n x^n$

(1) It converges ptwise in $(-R,R)$ where $R = \frac{1}{\lim_{n \to \infty} |c_n|^{\frac{1}{n}}}$ is the conv. radius.

(2) On $-R$ and R, the series might not converge.

(3) It converges uniformly in $[-R+\varepsilon, R-\varepsilon]$

(4) $f(x) := \sum_{n=0}^{\infty} c_n x^n$ is cts & diffable on $(-R,R)$ and $f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$ on $(-R,R)$

(5) f is smooth

(6) If $R = \infty$ then f is called analytic

(7) If $\sum_{n=0}^{\infty} c_n R^n$ converges then $\lim_{x \to R^-} f(x) = \sum_{n=0}^{\infty} c_n R^n$

Analyticity in power series: Given $f(x) := \sum_{n=0}^{\infty} a_n x^n$ and $g(x) := \sum_{n=0}^{\infty} b_n x^n$ on $S := (-R,R)$, and $E := \{x \in S \mid f(x) = g(x)\}$, if E has limit pt then $f = g$ on S.

Abel's thm: If $f(x) := \sum_{k=0}^{\infty} a_k x^k$ has radius of conv 1 and $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{x \to 1^-} f(x) = \sum_{k=0}^{\infty} a_k$.

Arzelà-Ascoli: Given $\{f_n\}$ cts on $[a,b]$, then: $\{f_n\}$ unif. bdd & equicts. ⟺ every subseq. of $\{f_n\}$ has unif. conv. subseq.